J Glob Optim (2010) 46:537–542
DOI 10.1007/s10898-009-9437-8

Vector equilibrium flows with nonconvex ordering relations

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Received: 5 May 2009 / Accepted: 7 May 2009 / Published online: 29 May 2009

Abstract In this note we introduce the concept of vector network equilibrium flows when the ordering cone is the union of finitely many closed and convex cones. We show that the set of vector network equilibrium flows is equal to the intersection of finitely many sets, where each set is a collection of vector equilibrium flows with respect to a closed and convex cone. Sufficient and necessary conditions for a vector equilibrium flow are presented in terms of scalar equilibrium flows.

Keywords Vector network equilibrium flow · Nonconvex ordering · Solution set · Vector variational inequality

1 Introduction

Following the introduction of Wardrop's principle [8] for equilibrium flows in traffic networks that considers only the minimum delay criterion, there have been generalizations of the principle that consider multiple criteria. Indeed, in the real world, people in choosing a path to travel consider not only minimum delay but also other factors such as cost, safety and convenience. Such generalizations have been considered by [1] and [3]. Under the symmetry

This paper was supported in part by The Hong Kong Polytechnic University under grant number G-YF17. Cheng was also supported by the Research Grants Council of Hong Kong under grant number PolyU 5135/06E, Li was also supported by the National Natural Science Foundation of China (Grant number: 60574073) and the Natural Science Foundation Project of CQ CSTC (Grant number: 2007BB6117) and Yang was also supported by the National Natural Science Foundation of China (Grant number: 10831009) .

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assumption for vector-valued cost functions, vector variational inequalities have been derived as a necessary optimality condition for a vector equilibrium flow. In those discussions, the ordering cone was assumed to be convex. The model of vector variational inequalities was introduced by [2].

Recently, [7] (See also [4]) considered a nonconvex ordering for vector optimization problems, which seeks to generalize existing results from a theoretical perspective. Following this trend, we introduce in this paper the concept of vector equilibrium flows when the ordering cone is the union of finitely many closed and convex cones. We show that this set of vector equilibrium flows is equal to the intersection of finitely many sets, where each set is a collection of vector equilibrium flows with respect to a closed and convex cone. We note that [4] have recently obtained such a relation between a vector variational inequality and a vector optimization problem. In this note we obtain sufficient and necessary conditions for vector equilibrium flows in terms of a scalar equilibrium flow.

2 A model for vector equilibria with nonconvex orderings

Consider a transportation network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ denotes the set of nodes and $\mathcal{A}$ the set of arcs. Let $\mathcal{I}$ be the set of origin-destination (O-D) pairs and P_i $(i \in \mathcal{I})$ denote the set of available paths joining the O-D pair i. For a given path $p \in P_i$, let h_p denote the traffic flow on this path and $h = [h_p] \in \mathbb{R}^M$, $M = \sum_{i \in \mathcal{I}} |P_i|$. A path flow vector h induces a flow v_a on each arc $a \in \mathcal{A}$ given by:

$$v_a = \sum_{i \in \mathcal{I}} \sum_{p \in P_i} \delta_{ap} h_p,$$

where $\delta_{ap} = 1$ if arc a belongs to path p and 0 otherwise. $\Delta = [\delta_{ap}] \in \mathbb{R}^{|\mathcal{A}| \times M}$ is the arc path incidence matrix. Let $v = [v_a] \in \mathbb{R}^{|\mathcal{A}|}$ be the vector of arc flows. We assume that the demand $d_i \geq 0$ of traffic flow is fixed for each O-D pair $i \in \mathcal{I}$. Let

$$\mathcal{H} = \left\{ h \mid h \geq_{\mathbb{R}^M_+} 0, \sum_{p \in P_i} h_p = d_i \ \forall i \in \mathcal{I} \right\}$$

be the feasible set.

Let $t_a(v) \in \mathbb{R}^r$ be the vector cost for arc a (including such cost elements as time delay, monetary cost and others) and be in general a function of all the arc flows. The (vector) cost $\tau_p \in \mathbb{R}^r$ along a path is assumed to be the sum of all the arc costs along this path; thus

$$\tau_p(h) = \sum_{a \in \mathcal{A}} \delta_{ap} t_a(v).$$

Vector equilibrium principles have been considered in the literature when the ordering cone is convex. In this note we discuss these principles when the ordering cone is not necessarily convex. In particular, we assume that the ordering cone is the union of finitely many closed and convex cones, with a nonempty interior. That is, $K = \bigcup_{j=1}^{l} K_j$ and $int K \neq \emptyset$, where each K_j $(j = 1, \ldots, l)$ is a nonempty, closed, pointed and convex cone of $\mathbb{R}^r$.

For simplicity, we adapt the following notation for the orderings. Given a closed cone $S \subset \mathbb{R}^r$ with $int S \neq \emptyset$, the pre-orderings $\leq_{S \setminus \{0\}}$, $\leq_{int S}$, $\nleq_{S \setminus \{0\}}$, and $\nleq_{int S}$ are defined as:

for $\xi, \eta \in \mathbb{R}^r$,

$$\xi \leq_{S\backslash\{0\}} \eta \Longleftrightarrow \eta - \xi \in S\backslash\{0\};$$
$$\xi \leq_{intS} \eta \Longleftrightarrow \eta - \xi \in intS;$$
$$\xi \not\leq_{S\backslash\{0\}} \eta \Longleftrightarrow \eta - \xi \notin S\backslash\{0\};$$
$$\xi \not\leq_{intS} \eta \Longleftrightarrow \eta - \xi \notin intS.$$

The pre-orderings $\geq_{S\backslash\{0\}}$, $\geq_{intS}$, $\not\geq_{S\backslash\{0\}}$ and $\not\geq_{intS}$ are defined similarly.

Definition 2.1 Given a flow $h \in \mathcal{H}$, we say that

(i) a path $p \in P_i$ for an O-D pair i is *efficient* if there does not exist another path p' such that $\tau_{p'}(h) \leq_{K\backslash\{0\}} \tau_p(h)$;
(ii) a path $p \in P_i$ for an O-D pair i is *weakly efficient* if there does not exist another path p' such that $\tau_{p'}(h) \leq_{intK} \tau_p(h)$.

Let $\Gamma_i(h) = \{\tau_p(h),\ p \in P_i\}$ denote the (discrete) set of vector costs for all the paths of O-D pair i. Let $\mathcal{I}_i^K(h) = \{k \in P_i \mid \tau_k(h) \not\geq_{K\backslash\{0\}} \tau_p(h),\ \forall p \in P_i\} \subseteq P_i$ and $\mathcal{WI}_i^K(h) = \{k \in P_i \mid \tau_k(h) \not\geq_{intK} \tau_p(h),\ \forall p \in P_i\} \subseteq P_i$ denote the index sets of all the efficient paths and weakly efficient paths for O-D pair i, respectively.

We define respectively the *efficient frontier* and *weak efficient frontier* for O-D pair i and K to be sets of efficient points and weak efficient points in the cost-space of O-D pair i as follows:

$$Min_K(\Gamma_i(h)) = \{\xi \in \mathbb{R}^r \mid \xi = \tau_p(h) \text{ where } p \in \mathcal{I}_i^K(h)\},$$
$$Min_{intK}(\Gamma_i(h)) = \{\xi \in \mathbb{R}^r \mid \xi = \tau_p(h) \text{ where } p \in \mathcal{WI}_i^K(h)\}.$$

The sets $\mathcal{I}_i^{K_j}(h)$, $Min_{K_j}(\Gamma_i(h))$, $\mathcal{WI}_i^{K_j}(h)$ and $Min_{intK_j}(\Gamma_i(h))$, $j = 1, \ldots, l$ are defined similarly.

In [1], a vector Wardrop's principle was proposed, where the ordering cone K is closed and convex. The following definition for vector Wardrop's principle doesn't assume the convexity of the ordering cone K.

Definition 2.2
(i) A path flow vector $h \in \mathcal{H}$ is said to be in *vector equilibrium for* K if

$$\forall i \in \mathcal{I}, \forall p, \bar{p} \in P_i,\ h_p = 0 \ \text{ whenever } \tau_p(h) - \tau_{\bar{p}}(h) \geq_{K\backslash\{0\}} 0. \tag{1}$$

(ii) A path flow vector $h \in \mathcal{H}$ is said to be in *weak vector equilibrium for* K if

$$\forall i \in \mathcal{I}, \forall p, \bar{p} \in P_i,\ h_p = 0 \ \text{ whenever } \tau_p(h) - \tau_{\bar{p}}(h) \geq_{intK} 0.$$

By $\mathcal{S}$ and $\mathcal{WS}$, we denote the sets of all the vector equilibrium flows and all the weak vector equilibrium flows for K, respectively. By $\mathcal{S}_j$ and $\mathcal{WS}_j$, we denote the sets of all the vector equilibrium flows and all the weak vector equilibrium flows for K_j, respectively. When $K = \mathbb{R}_+^r$, the above definitions reduce to the ones in [3].

Theorem 2.1 *We have the following results:*

(i) $\mathcal{S} = \bigcap_{j=1}^{l} \mathcal{S}_j$;
(ii) $\mathcal{WS} = \bigcap_{j=1}^{l} \mathcal{WS}_j$.

Proof

(i) Suppose that $h \in \mathcal{S}$ is a vector equilibrium flow for K. For each fixed j $(j = 1, \ldots, l)$, we want to prove $h \in \mathcal{S}_j$. Let $i \in \mathcal{I}$ and $p, \bar{p} \in P_i$. Suppose that $\tau_p(h) - \tau_{\bar{p}}(h) \geq_{K_j \setminus \{0\}}$. From the definition of K, it is also true that $\tau_p(h) - \tau_{\bar{p}}(h) \geq_{K \setminus \{0\}} 0$. It follows from Eq. (1) that $h_p = 0$. Thus h is a vector equilibrium flow with respect to K_j, i.e., $h \in \mathcal{S}_j$. Thus $h \in \bigcap_{j=1}^{l} \mathcal{S}_j$. So $h \in \bigcap_{j=1}^{l} \mathcal{S}_j$.
Conversely, suppose that h is a vector equilibrium flow with respect to each K_j, $j = 1, \ldots, l$. Let $i \in \mathcal{I}$ and $\forall p, \bar{p} \in P_i$. Suppose that $\tau_p(h) - \tau_{\bar{p}}(h) \geq_{K \setminus \{0\}} 0$. From the definition of K, there exists a j_0 such that $\tau_p(h) - \tau_{\bar{p}}(h) \geq_{K_{j_0} \setminus \{0\}} 0$. By the definition of $h \in K_{j_0}$, $h_p = 0$. So h is also a vector equilibrium flow with respect to K, i.e., $h \in \mathcal{S}$. Thus $\bigcap_{j=1}^{l} \mathcal{S}_j \subset \mathcal{S}$. Thus, (i) holds.

(ii) The proof is similar to that of (i), but replacing $K \setminus \{0\}$ and $K_j \setminus \{0\}$ by int K and int K_j, respectively. □

Remark 2.1 As a consequence of Theorem 2.1, the existence of a (weak) vector equilibrium flow when the ordering cone is a union of some convex cones is equivalent to that all subproblems (each with a convex cone) have a common one, while the existence of such a flow for each subproblem can be obtained using the existence result of a scalar equilibrium flow (see [5]) and the relation between a vector equilibrium flow and a scalar one (see [3]).

Now we give an example to illustrate the result of Theorem 2.1.

Example 2.1 Consider a network flow problem that consists of two nodes x and y, two arcs a and b, and a single O-D pair $w = (x, y)$. Assume the travel demand for w is $d_w = 4$. The path cost functions from $\mathbb{R}^2$ to $\mathbb{R}^3$ are, respectively,

$$\tau_{p_1}(h) = (h_{p_1}, h_{p_1}, h_{p_1}) \quad \text{and} \quad \tau_{p_2}(h) = (h_{p_2}, h_{p_2}, 2h_{p_2}),$$

where p_1 and p_2 denote two paths, which consist of arcs a and b, respectively. Suppose that $K = \bigcup_{j=1}^{3} K_j$, where K_1, K_2 *and* K_3 are given by

$$K_1 = \{(x, y, z) \in \mathbb{R}^3 \mid (2x - 1.5z)^2 + (2y - z)^2 \leq z^2, x \geq 0, y \geq 0, z \geq 0\},$$
$$K_2 = \{(x, y, z) \in \mathbb{R}^3 \mid (2x - z)^2 + (2y - (1 + \sqrt{3}/2)z)^2 \leq z^2, x \geq 0, y \geq 0, z \geq 0\},$$

and

$$K_3 = \{(x, y, z) \in \mathbb{R}^3 \mid (2x - 2z)^2 + (2y - (1 + \sqrt{3}/2)z)^2 \leq z^2, x \geq 0, y \geq 0, z \geq 0\}.$$

Naturally, K is a nonconvex cone. From Definition 2.2 (i), we have that the sets of all the vector equilibrium flows for K_1, K_2 and K_3 are, respectively,

$$S_1 = \{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4 \text{ and } h_{p_2} < h_{p_1}\} \bigcup \{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4, h_{p_2} \geq h_{p_1} \text{ and } 0.25h_{p_1}^2 + 5h_{p_1}h_{p_2} > 3h_{p_2}^2\},$$

$$S_2 = \{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4 \text{ and } h_{p_2} < h_{p_1}\} \bigcup \{(h_{p_1}, h_{p_2}) \in \mathcal{R}^2_+ \mid h_{p_1} + h_{p_2} = 4, h_{p_2} \geq h_{p_1} \text{ and } (1 - \sqrt{3}/2)^2 h_{p_1}^2 + (2\sqrt{3} + 1)h_{p_1}h_{p_2} > h_{p_2}^2\},$$

$$S_3 = \{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4 \text{ and } h_{p_1} < 2h_{p_2}\} \bigcup \{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4, h_{p_1} \geq 2h_{p_2} \text{ and } (3/4 - \sqrt{3})h_{p_1}^2 + (2\sqrt{3} + 1)h_{p_1}h_{p_2} + 3h_{p_2}^2 > 0\}.$$

Thus, we get

$$
\begin{aligned}
S &= \bigcap_{j=1}^{3} S_j \\
&= \{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4 \text{ and } h_{p_2} < h_{p_1} < 2h_{p_2}\} \bigcup \\
&\quad \{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4, h_{p_1} \geq 2h_{p_2} \\
&\qquad \text{and } (3/4 - \sqrt{3})h^2_{p_1} + (2\sqrt{3}+1)h_{p_1}h_{p_2} + 3h^2_{p_2} > 0\} \bigcup \\
&\quad \Big\{(h_{p_1}, h_{p_2}) \in \mathbb{R}^2_+ \mid h_{p_1} + h_{p_2} = 4, h_{p_2} \geq h_{p_1}, 0.25h^2_{p_1} + 5h_{p_1}h_{p_2} > 3h^2_{p_2} \\
&\qquad \text{and } (1 - \sqrt{3}/2)^2 h^2_{p_1} + (2\sqrt{3}+1)h_{p_1}h_{p_2} > h^2_{p_2}\Big\}.
\end{aligned}
$$

3 Sufficient and necessary conditions for vector equilibrium flows

For each K_j, $j = 1, \dots, l$, we define its dual cone as

$$K^*_j = \{\lambda = (\lambda_1, \dots, \lambda_r)^T \in \mathbb{R}^r \mid \lambda^T x \geq 0, \forall x \in K_j\}.$$

Let a parametric $\lambda \in K^*_j \backslash \{0\}$ be given. In [3], a path flow vector h is in λ-equilibrium for K_j if,

$$\forall i \in \mathcal{I},\ \forall p \in P_i,\ h_p = 0 \ \textit{whenever} \ \exists\, e_i \in Min_{K_j}(\Gamma_i(h)),\ \textit{such that}\ \lambda^\top \tau_p(h) > \lambda^\top e_i.$$

Theorem 3.1

(i) *If there exists $\lambda^j \in int K^*_j$, $j = 1, \dots, l$ such that h is in λ^j-equilibrium for each K_j, then h is in vector equilibrium for K.*
(ii) *If there exists $\lambda^j \in K^*_j \backslash \{0\}$, $j = 1, \dots, l$, such that h is in λ^j-equilibrium for each K_j, then h is in weak vector equilibrium for K.*

Proof

(i) Since $\lambda^j \in int K^*_j$, it follows from Theorem 4 in [9] that h is in vector equilibrium flow for K_j. By Theorem 2.1 (i), h is in vector equilibrium flow for K.
(ii) Similarly from Theorem 4 in [9], we can conclude that h is in weak vector equilibrium for K_j. By Theorem 2.1 (ii), h is in weak vector equilibrium flow for K. □

For $\lambda \in K^*_j \backslash \{0\}$, we define the minimum scalarized cost for O-D pair i as:

$$u_i(\lambda) = \min_{p \in P_i} \lambda^\top \tau_p(h).$$

Let $\lambda^j \in \text{int } K^*_j$. Following the proof of Lemma 2.1 in [3], we have, for some $e_i \in Min_{K_j}(\Gamma_i(h))$,

$$u_i(\lambda) = \lambda^\top e_i.$$

Thus, for $\lambda \in \text{int } K^*_j$, h is in λ-equilibrium for K_j if and only if the following condition holds:

$$\forall i \in \mathcal{I}, \forall p \in P_i,\ h_p = 0 \quad \text{whenever } \lambda^\top \tau_p(h) > u_i(\lambda).$$

Let a parameter $\lambda \in K_j^* \backslash \{0\}$ be given. Since an λ-equilibrium flow is based on a scalar cost, it follows from [5] that h is in λ-equilibrium for K_j if and only if h satisfies, for all $i \in \mathcal{I}$, $p \in P_i$,

$$(\lambda^\top \tau_p(h) - u_i(\lambda))h_p = 0, \tag{2}$$

$$\lambda^\top \tau_p(h) - u_i(\lambda) \geq 0, \tag{3}$$

$$\sum_{p \in P_i} h_p - d_i = 0, \tag{4}$$

$$h \geq 0, u_i(\lambda) \geq 0. \tag{5}$$

Thus, from Theorem 3.1, we have the following result.

Theorem 3.2

(i) *If there exist some $\lambda^j \in int K_j^*$, $j = 1, \ldots, l$, such that h satisfies Eqs. (2–5) where $\lambda = \lambda^j$, for all $i \in \mathcal{I}$, $p \in P_i$, then h is in vector equilibrium for K.*

(ii) *If there exist some $\lambda^j \in K_j^* \backslash \{0\}$, $j = 1, \ldots, l$, such that h satisfies Eqs. (2–5) with $\lambda = \lambda^j$, for all $i \in \mathcal{I}$, $p \in P_i$, then h is in weak vector equilibrium for K.*

Corollary 3.1 *If there exists some $\lambda \in (\sum_{j=1}^l K_j)^* \backslash \{0\}$ such that h satisfies Eqs. (2–5), for all $i \in \mathcal{I}$, $p \in P_i$, then h is in weak vector equilibrium for K.*

Proof Since $K_1, \ldots, K_l$ are nonempty and convex cones, it follows from Corollary 16.4.2 in [6] that

$$\left(\sum_{j=1}^{l} K_j \right)^* = \cap_{j=1}^l K_j^*.$$

By Theorem 3.2 (ii), the result holds. □

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